J. DIFFERENTIAL GEOMETRY
7 (1972) 535-548

TIGHT EQUIVARIANT IMBEDDINGS OF SYMMETRIC SPACES

EDMUND F. KELLY

The purpose of this work is to prove the results announced in [4].

Roughly speaking, an immersion $f: M^n \to \boldsymbol{R}^N$ of a compact connected manifold in $\boldsymbol{R}^N$ is tight if the height functions have the minimum number of critical points. A detailed treatment of total absolute curvature and tightness can be found in Chern-Lashof [2] and Kuiper [9], [10].

In [8] Kobayashi and Takeuchi have exhibited the existence of tight equivariant immersions for a large class of homogeneous spaces which contains many symmetric spaces (see also [5]). In this paper we show that they have effectively constructed all tight imbeddings of irreducible symmetric spaces.

The layout is as follows: In §§ 1 and 2 we develop notation and a formula for the second fundamental form of an equivariant immersion. In § 3 we prove a connection between the second fundamental form and 0-tightness, and deal with 0-tight submanifolds of the sphere. In §§ 4–7 we apply these results to symmetric spaces. Then in § 8 we tie 0-tight immersions of irreducible homogeneous spaces with minimal immersions in the sphere.

The results in this work are a slight generalization and expansion of some results contained in the author's thesis, however the method of proof is less algebraic than that announced in [4]. The author would like to thank his advisor, Professor S. Helgason, to whom he owes a debt of gratitude. Any unexplained terms in Lie Theory will be found in [3].

1. The second fundamental form

Let $f: M^n \to N$ be a C^∞ immersion of a manifold M^n in a Riemannian manifold N. We will not differentiate between $x \in M$ and its image $f(x)$ in N. Let $N_x = M_x \oplus M_x^\perp$ be the orthogonal decomposition of the tangent space N_x at $x \in M$, under the Riemannian metric on N. Let $\bar{\nabla}$ denote covariant differentiation on N. It is convenient for our purposes to follow [7] and use the following definition of the second fundamental form. If X and Y are vector fields on

Received April 29, 1971, and, in revised form, August 16, 1972. While at graduate school the author was supported for some time by a Postgraduate Studentship of the Ministry of Education of N. Ireland; preparation of this paper was supported by N.R.C.A. 7846.

M, then the second fundamental form of the immersion at x is the symmetric bilinear map $\alpha_x\colon M_x \times M_x \to M_x^\perp$ given by

$$\alpha_x(X_x, Y_x) = \text{normal component of } (\bar{\nabla}_X Y)_x .$$

(Where there is no danger of confusion we write α for α_x.)

If $(\nabla_X Y)_x = (\bar{\nabla}_X Y)_x - \alpha_x(X_x, Y_x)$, then ∇ is the unique Riemannian connection on M in the induced metric. If ξ is a normal vector field and X a vector field on M, let

$$-A_{\xi_x}(X_x) = \text{tangential component of } (\bar{\nabla}_X \xi)_x .$$

Then $A_{\xi_x}(X_x)$ is a symmetric linear operator $M_x \to M_x$, and $(A_\xi X, Y) = (\xi, \alpha(X, Y))$.

If in particular $N = R^{n+N}$, then for each $v \in R^{n+N}$ we define the height function ϕ_v on M by $\phi_v(x) = (v, f(x))$. Making the usual identification between R^{n+N} and its tangent space at any point we can for each $v \in R^{n+N}$ construct the following vector fields on M:

$$V = \operatorname{grad} \phi_v , \qquad \bar{V} = v - V .$$

Obviously V_x is the tangential component of v at x, and $\bar{V}$ the normal component. The family of vector fields constructed in this way has the following properties:

Proposition 1.1. *If v and w are in R^{n+N} and M is immersed in R^{n+N}, then*

(i) $\nabla_X V = A_{\bar{V}} X$,

(ii) $[V, W] = A_{\bar{W}} V - A_{\bar{V}} W$,

(iii) $[X, V] = \operatorname{grad}(X\phi_v)$, *for a killing vector field X on M.*

Proof. (i) $\nabla_X V = (\bar{\nabla}_X V)^\tau = (\bar{\nabla}_X v - \bar{\nabla}_X \bar{V})^\tau = -(\bar{\nabla}_X \bar{V})^\tau = A_{\bar{V}} X$.

(ii) follows immediate from (i) since ∇ has zero torsion.

(iii) If X is a Killing vector field and Y is any vector field, then

$$\begin{aligned}([X, V], Y) &= (\nabla_X V, Y) - (\nabla_V X, Y) = (A_{\bar{V}} X, Y) + (V, \nabla_Y X) \\ &= (X, A_{\bar{V}} Y) + (V, \nabla_Y X) = (X, \nabla_Y V) + (V, \nabla_Y X) \\ &= Y(X, V) = Y(X\phi_v) ,\end{aligned}$$

so $[X, V] = \operatorname{grad}(X\phi_v)$.

2. Immersions of compact homogeneous spaces

Let G/H be an isotropy irreducible compact homogeneous space, and $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ the standard decomposition of the Lie algebra $\mathfrak{g}$ of G. We shall always assume that G/K carries the G-invariant Riemannian structure induced by $-B$ on $\mathfrak{m}$ where B is the Killing form on $\mathfrak{g}$.

Let $\pi\colon G \to \operatorname{End}(E^N)$ be a nontrivial real class-one representation of G with nonzero H-fixed vector e. Then π induces a map, also denoted by π, from G/H into E^N by $\pi(gH) = \pi(g)e$. We always assume representations are orthogonal.

Let π also denote the representation of $\mathfrak{g}$ induced by the representation of G. Then the map $gH \to \pi(g)e$ gives an equivariant immersion of G/H. Equivariance is obvious, the fact that it is an immersion comes from the irreducibility of the action of H on $\mathfrak{m}$, for if X, Y are in $\mathfrak{m}$ let $\langle\langle X, Y\rangle\rangle = (\pi(X)e, \pi(Y)e)$ (the Euclidean inner product on E^N), then $\langle\langle\ ,\ \rangle\rangle$ is H invariant and there is only one such up to scalar multiple.

Remark. This leads to the following lower bound for the dimensions of representations which the author imagines is well known, but he has not seen it remarked in the literature.

Lemma 2.1. *If G/K is a compact irreducible symmetric space, and π: $G \to \mathrm{End}\,(E^N)$ a real class one representation, then*

$$N \geq \mathrm{rank}\,(G/K) + \dim\,(G/K)\ .$$

Proof. Immediate consequence of the following theorem of Chern and Kuiper [1] and Otsuki [13]: If M^n is a compact Riemannian submanifold of R^{n+p}, and at every $x \in M$ the tangent space M_x contains an m-dimensional subspace with the sectional curvature of any plane in the subspace ≤ 0, then $p \geq m$. q.e.d.

We now calculate the second fundamental form of the immersion π: $G/H \to E^N$. To do so we shall use two lemmas.

Lemma 2.2. *Let f: $M \to R^N$ be an immersion. Suppose $\{x_1, \cdots, x_n\}$ is a local coordinate system on a neighborhood U of m in M. Then*

$$\alpha\left(\left(\frac{\partial}{\partial x_i}\right)_m, \left(\frac{\partial}{\partial x_j}\right)_m\right) = \textit{normal component of } \left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)_m$$

under the usual identification of R^N and its tangent space.

Proof. Cf. [7, pp. 17 and 18].

Lemma 2.3. *Let G/H be isotropy irreducible, and π an orthogonal representation of G with H fixed vector e. Then*

$$(\pi(X)e, \pi(Y)\pi(Z)e + \pi(Z)\pi(Y)e) = 0 \qquad \textit{all } X, Y, Z \in \mathfrak{m}\ .$$

Proof. Assert first $(\pi(X)e, \pi(Y)\pi(Y)e) = 0$ all $X, Y \in \mathfrak{m}$. $(\pi(X)\pi(Y)e, \pi(Y)e) = 0$ since the representation is orthogonal, but

$$\begin{aligned}(\pi(X)\pi(Y)e, \pi(Y)e) &= (\pi[X, Y]e, \pi(Y)e) + (\pi(Y)\pi(X)e, \pi(Y)e) \\ &= -B([X, Y], Y) - (\pi(X)e, \pi(Y)\pi(Y)e) \\ &= -(\pi(X)e, \pi(Y)\pi(Y)e)\ .\end{aligned}$$

Now

$$\pi(Z + Y)\pi(Z + Y)e = \pi(Z)\pi(Z)e + \pi(Y)\pi(Y)e + \pi(Z)\pi(Y)e + \pi(Y)\pi(Z)e .$$

So the lemma now follows easily.

Theorem 2.1. *Let α be the second fundamental form of the immersion $\pi: G/H \to E^N$. Then at the origin of G/H*

$$\alpha(X, Y) = \tfrac{1}{2}(\pi(X)\pi(Y)e + \pi(Y)\pi(X)e) \qquad \text{all } X, Y \in \mathfrak{m} .$$

Proof. Let $X_1, \cdots, X_n$ be an orthonormal basis for $\mathfrak{m}$. Let U be a normal neighborhood of 0 in G/H so that $\operatorname{Exp}(x_1X_1 + x_2X_2 + \cdots) \to (x_1, \cdots)$ is a coordinate system about 0 with

$$(\partial/\partial x_i)_0 = X_i ,$$
$$\begin{aligned}\pi(\operatorname{Exp}(x_1X_1 + \cdots + x_nX_n) &= \pi(\exp(x_1X_1 + \cdots + x_nX_n)e \\ &= \exp(x_1\pi(X_1) + \cdots + x_n\pi(X_n))e .\end{aligned}$$

To compute $\partial^2\pi/\partial x_i\partial x_j\ (0, 0, \cdots)$ we need only to consider $\exp(x_i\pi(x_i) + x_j\pi(X_j))e$. But if A and B are any $r \times r$ matrices, then

$$\frac{\partial^2}{\partial t_1\partial t_2}(\exp(t_1A + t_2B))|_{(0,0)} = \frac{1}{2}(AB + BA) .$$

(This can be proven by expansion in series.) Hence

$$\left.\frac{\partial^2\pi}{\partial x_i\partial x_j}\right|_0 = \frac{1}{2}(\pi(X_i)\pi(X_j)e + \pi(X_j)\pi(X_i)e) ,$$

which by Lemma 2.3 is normal, so

$$\alpha(X_i, X_j) = \tfrac{1}{2}(\pi(X_i)\pi(X_j)e + \pi(X_j)\pi(X_i)e) .$$

For any X, Y in $\mathfrak{m}$ the bilinearity of α gives

$$\alpha(X, Y) = \tfrac{1}{2}(\pi(X)\pi(Y)e + \pi(Y)\pi(X)e) .$$

Corollary. *If G/H is symmetric, then $\alpha(X, Y) = \pi(X)\pi(Y)e$.*

Proof. Since G/H is symmetric, we have $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$ and therefore $\pi[X, Y]e = 0$ for all X and Y in $\mathfrak{m}$, so that $\pi(X)\pi(Y)e = \pi(Y)\pi(X)e$.

Remark. For symmetric spaces the proof of Lemma 2.3 does not need irreducibility of G/H, so the formula is general.

3. 0-tight submanifolds of the sphere

First let us recall the Morse inequalities. If M^n is a compact connected manifold, and $\phi: M^n \to R^1$ is a smooth function, then ϕ is said to be nondegenerate if and only if all its critical points are nondegenerate.

We define for nondegenerate ϕ:

$$\mu_k(\phi) = \text{number of critical points of index } k \text{ of } \phi\ , \qquad \mu(\phi) = \sum_k \mu_k(\phi)\ ,$$

$$\mu_k(M^n) = \inf_{\phi \text{ nondegenerate}} \mu_k(\phi)\ , \qquad \mu(M) = \inf_{\phi \text{ nondegenerate}} \mu(\phi)\ .$$

A nondegenerate function ϕ is said to be tight if $\mu(\phi) = \mu(M^n)$, and to be 0-tight if $\mu_0(\phi) = 1$. By M. Morse [12] ϕ is 0-tight if it is tight. We have the Morse inequalities [11]:

$$\mu_k(M^n) \geq \dim H_k(M, *) \qquad \text{for any coefficient field}\ .$$

If $f: M^n \to R^{n+N}$ is an immersion, then by Sard's theorem the height function $\phi_v(x)$ has only nondegenerate critical points for almost all (with respect to Euclidean measure) $v \in R^{n+N}$.

We say the immersion f is tight (0-tight) if ϕ_v is tight (0-tight) for nondegenerate height functions. The property of being tight (0-tight) is invariant under affine transformations of R^{n+N}. The immersion f is said to be substantial if $f(M^n)$ is not contained in a hyperplane of R^{n+N}.

The following theorem is inspired by Theorem 4, [9].

Theorem 3.1. *If $f: M^n \to R^{n+N}$ is a substantial 0-tight immersion of a compact connected manifold, then there is an open set U of M such that $\alpha: M_x \otimes M_x \to M_x^\perp$ is an onto map for all $x \in U$.*

Proof. Let $v \in R^{n+N}$ be such that ϕ_v is nondegenerate. Since tightness is a translation invariant, we can assume $\phi_v(x)$ attains its minimum at x_0 where $f(x_0) = 0$. Then $\phi_v(x)$ has a nondegenerate critical point of index at x_0.

If α_{x_0} is not onto, then by Lemma 2.2 we can choose $0 \neq z \in M_{x_0}^\perp$ such that $F_z(x) = (v + z, f(x))$ has a nondegenerate critical point of index 0 at x_0.

We can find λ such that $(v + \lambda t, f(x))$ assumes both $+ve$ and $-ve$ values. In fact, the function $h(x) = (z, f(x))/(v, f(x))$ is not constant since f is substantial. Thus there is λ such that h takes values $> 1/\lambda$ and $< -1/\lambda$. Hence the function $(z - \lambda v, f(x))$ assumes $+ve$ and $-ve$ values on M.

Let $w = \lambda z - a$. Then $\phi_w(x) = (w, f(x))$ has a nondegenerate critical point of index 0 at x_0 and $\phi_w(x_0) = 0$. Assert we can choose w' in R^{N+n} such that $\phi_{w'}(x)$ has nondegenerate critical points and $\phi_{w'}$ has a critical point of index 0 near x_0 which is not a true minimum.

There is a local coordinate system $(u_1, \cdots, u_n)$ on an open neighborhood U of $x_0 = (0, \cdots, 0)$ such that

$$\phi_w(x) = u_1^2 + \cdots + u_n^2\ .$$

Consider the "sphere" $S(r) \subset U$ given by $u_1^2 + \cdots + u_n^2 = r^2$, $\phi_w(x) = r^2$ on

$S(r)$. We can choose w' in any neighborhood of w such that $\phi_{w'}$ has only nondegeneratecritical points.

$$\|\phi_w - \phi_{w'}\|_M = \|(f(x), w - w')\|_M \leq A \|w - w'\| ,$$

where $A = \max_{x \in M} \|f(x)\|$. Choose w' with $\|w - w'\| < \frac{1}{2}r^2/A$. Then

$$\|\phi_w - \phi_{w'}\| \leq \tfrac{1}{2}r^2 .$$

$\phi_{w'}$ has a minimum in the closed ball $\bar{S}(r)$. We assert that this minimum does not occur on the sphere $S(r)$. But this is easy to prove since $\phi_{w'}(x_0) = 0$, and $\phi_{w'}(x) > \frac{1}{2}r^2$ for $x \in S(r)$. So the minimum on $\bar{S}(r)$ is in fact a critical point of index 0 of $\phi_{w'}$. Since ϕ_w takes $+ve$ and $-ve$ values, we can clearly choose w' such that we will have its absolute minimum outside $S(r)$.

So we have constructed a nondegenerate height function with two critical points of index 0, contradicting 0-tightness. Hence α is onto at x_0. The fact that α is onto in an open neighborhood of x_0 is a trivial consequence of the differentiability of α.

Corollary 1 (cf. [9, Theorem 4]). *If $f: M^n \to R^{n+N}$ is a 0-tight immersion, then $N \leq \frac{1}{2}n(n + 1)$.*

Proof. Trivial since α is a symmetric map from $M_x \otimes M_x$ to $M_x^{\perp}$.

Corollary 2. *If $f: M^n \to S^{n+N-1}$ is an immersion in the sphere which is also a 0-tight immersion in R^{n+N}, then the second fundamental form is onto everywhere.*

Proof. Let $f(x_0) = a$. Then the function $\phi_{-a}(x)$ has a nondegenerate critical point of index 0 at x_0, for as is well known the operator $A_a = -$Identity on M_{x_0}. Since in the proof of Theorem 3.1 we use only the nondegeneracy of the minimum, everything can be carried over. q.e.d.

We now have a theorem which as well as being independently interesting is important in the classification of tight equivariant imbeddings.

Theorem 3.2. *If $f: M^n \to S^{n+N-1}$ is an immersion of a compact connected manifold M in the sphere, and f is 0-tight when considered as an immersion in R^{n+N}, then f is in fact an imbedding.*

Proof. Suppose $f(x_1) = f(x_2) = a$. Then ϕ_{-a} has a critical point of index 0 at x_1 and x_2. We can, by following the lines of the proof of Theorem 3.1, construct a nondegenerate height function ϕ_v, with two critical points of index 0, one near x_1 and the other near x_2.

Remark. Theorem 3.2 proves a fortiori a conjecture of Wilson [14].

4. Reduction of the problem for homogeneous spaces

We now prove a theorem which for isotropy irreducible homogeneous spaces reduces the problem considerably.

Theorem 4.1. *Let G/H be a compact homogeneous space, and π a class-one real representation of G giving a 0-tight substantial imbedding $\pi: G/E \to E_\pi$. If π is reducible, $\pi = \rho + \mu$, and ρ gives an immersion $\rho: G/H \to E_\rho$, then $\rho: G/H \to E_\rho$ is 0-tight.*

Before proceeding with the proof we recall the two-piece property of [10]. An immersion $f: M \to E^N$ has the two-piece property if given any hyperplane $H \subset E^N$, $\{m \in M \mid f(m) \notin H\}$ has at most two components.

Lemma 4.1 [10]. *Let $f: M \to E^N$ be an immersion. Then f is 0-tight if and only if it has the two-piece property.*

Proof. Op. Cit.

Proof of Theorem 4.1. Let E_ρ and E_μ be the representation spaces for ρ and μ respectively. Since $\pi: G/H \to E_\pi$ is substantial, then the H-fixed vector e can be written $e = e_\rho + e_\mu$, where $0 \neq e_\rho$ is H-fixed in E_ρ and $0 \neq e_\mu$ is H-fixed in E_μ.

Supopse ρ gives an immersion of G/H in E_ρ. We show ρ satisfies the two-piece property.

Since π satisfies the two-piece property, given any $v \in E_\pi, \{\rho \in G/H \mid (\pi(\rho), v) \neq c\}$ has at most two components for any constant c. Write $v = v_\rho + v_\mu$, $v_\rho \in E_\rho$ and $v_\mu \in E_\mu$, and $\rho = g \cdot 0$. Then $(\pi(g)e, v) = (\rho(g)e_\rho, v_\rho) + (\mu(g)e_\mu, v_\mu)$. If in particular we consider $v_\mu = 0$, then for any $v_\rho \in E_\rho$, $\{gH \in G/H \mid (\rho(g)e_\rho, v_\rho) \neq c\}$ has at most two components. So $\rho: G/H \to E_\rho$ satisfies the two-piece property, and hence is 0-tight by Lemma 4.1.

Corollary. *Suppose G/H is an isotropy irreducible compact homogeneous space, and π a real class-one representation of G such that the immersion $\pi: G/H \to E_\pi$ is 0-tight. Then there is an irreducible class-one representation π' of G such that $\pi': G/H \to E_{\pi'}$ is 0-tight.*

Proof. There is no loss in generality in assuming $\pi: G/H \to E_\pi$ is substantial, for if not there is a vector v with $(v, \pi(g)e) = 0$ for all $g \in G$, where e is H-fixed, so there is a G-invariant space E_v with $(E_v, G/H) = 0$.

Suppose $E_\pi = E_\rho + E_\mu$. Since the immersion is substantial, we can write, as in Theorem 4.1, $e = e_\rho + e_\mu$. Thus as in the preamble to § 2 for all $x \in \mathfrak{m}$ (where $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$), $\pi(x)e_\rho = 0$ or $\pi(x)e_\rho \neq 0$, so either ρ or μ gives an immersion. Suppose ρ gives an immersion. We repeat the process on E_ρ, and eventually we get an immersion $\pi': G/H \to E_{\pi'}$ where π' is irreducible. Then repeated applications of Theorem 4.1 show π' is 0-tight.

Remark. Henceforth we assume all representations are irreducible unless explicitly stated otherwise.

5. Symmetric *R*-spaces

The theory of R-spaces is somewhat scattered throughout the literature, so in this section we organize what we need. We do not define a general R-space but give a somewhat ad-hoc definition of symmetric R-spaces, which is sufficient for our purposes.

Let $\mathfrak{L}$ be a real semi-simple Lie algebra, and $Z \in \mathfrak{L}$ be such that ad Z is semi-simple with real eigenvalues 0, ± 1.

Theorem 5.1. *There is a Cartan decomposion* $\mathfrak{L} = \mathfrak{g} + \mathfrak{p}$ *such that* $Z \in \mathfrak{p}$.

Proof. Cf. [6, Theorems 2 and 3].

Let $\mathfrak{L} = \mathfrak{L}_{-1} + \mathfrak{L}_0 + \mathfrak{L}_1$ be the eigenspace decomposition of $\mathfrak{L}$ relative to ad Z, and define $\alpha: \mathfrak{L} \to \mathfrak{L}$ to be a linear map by $\alpha(X + Y + W) = -X + Y - W$, $X \in \mathfrak{L}_{-1}$, $Y \in \mathfrak{L}_0$, $W \in \mathfrak{L}_1$. Then α is an involutive automorphism.

If $\mathfrak{L} = \mathfrak{g}' + \mathfrak{p}'$ is a Cartan decomposition with involution σ', then $B^{\sigma'}(X, Y) = -B_L(X, \sigma' Y)$ is a symmetric positive definite bilinear form on $\mathfrak{L}$, and $\alpha\sigma'$ is self-adjoint with respect to $B^{\sigma'}$. Thus $\rho = (\alpha\sigma')^2$ can be diagonalized with positive entries so as in the proof of [3,Theorem 7.1, p. 156] we can define ρ^τ to be a one-parameter group of automorphisms of $\mathfrak{L}$ and $\sigma = \rho^{1/4}\sigma'\rho^{-1/4}$ is a Cartan involution of $\mathfrak{L}$ which commutes with α. Let $L = \mathfrak{g} + \mathfrak{p}$ be the corresponding Cartan decomposition of L.

Thus $\mathfrak{L}_0 = \mathfrak{L}_0 \cap \mathfrak{g} + \mathfrak{L}_0 \cap \mathfrak{p}$, a direct sum. To see that $Z \in \mathfrak{p}$ it suffices to show that ad Z is symmetric with respect to B^σ. q.e.d.

Now let (L, G) be a pair associated with $(\mathfrak{L}, \sigma)$ such that L has no center.

Theorem 5.2. *Let* $K = \{g \in G \mid \mathrm{ad}_L\, gZ = Z\}$. *Then*

(i) G/K *is symmetric,*

(ii) *the immersion* $\phi: G/K \to \mathfrak{p}$ *by* $\phi(gK) = \mathrm{ad}_L\, gZ$ *is tight and equivariant.*

Proof. (i) This line is indicated in [8]. Since L has no center, $\mathrm{Ad}_L: L \to \mathrm{Int}(\mathfrak{L})$ is an analytic isomorphism onto, so we shall assume $L = \mathrm{Int}(\mathfrak{L})$. Let L^c be the complex Lie group $\mathrm{Int}(\mathfrak{L}^c)$ where $\mathfrak{L}^c$ is the complexification of $\mathfrak{L}$. Then $L \subset L^c$, and $\exp(i\pi Z) \in L^c$ where $i = \sqrt{-1}$.

If θ is the inner automorphism of L^c defined by $\exp(i\pi Z)$, then $\theta^2 = \mathrm{Id}$, and G is θ-stable. For by calculation one can show L is stable in L_c under Ad_{L^c} $(\exp i\pi Z)$, and then $\sigma_0\, \mathrm{Ad}(\exp i\pi Z) = \mathrm{Ad}(\exp i\pi Z) 0 \sigma$.

If $\theta|G$ is also denoted by θ, then $(K_\theta)_0 \subset K_\theta$ where $(K_\theta)_0$ is the connected component of the identity in K_θ, so G/K is Riemannian symmetric.

(ii) The equivariance of ϕ is obvious.

Tightness is proven in [8] for an even more general type of space.

Definition. A homogeneous space G/K is a symmetric R-space if it can be constructed as in Theorems 5.1 and 5.2.

6. The fundamental lemma

We now examine the implications of Theorem 3.1 for equivariant immersions of symmetric spaces.

Let G/K be an irreducible symmetric space, and π an irreducible class-one representation of G with K-fixed vector e giving the immersion $\pi: G/K \to E^N$. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the standard decomposition of the Lie algebra of $\mathfrak{g}$. If the second fundamental form is onto, we have

$$E^N = T_0 + T_0^\perp .$$

where

$$T_0 = \{\pi(X)e \mid X \in \mathfrak{p}\} , \qquad T_0^\perp = \text{linear hull of } \{\pi(X)\pi(X)e \mid X \in \mathfrak{p}\} .$$

We shall need the following lemma.

Lemma 6.1. *If π is a real orthogonal representation of $\mathfrak{g}$ with vector $e \neq 0$ annihilated by $\mathfrak{k}$, then*

$$(\pi(X)\pi(X)e, \pi(Z)\pi(Y)\pi(Y)e) = 0 \qquad \text{all } X, Y, Z \text{ in } \mathfrak{p}.$$

Proof. We know

$$(\pi(Z)\pi(X)\pi(Y)e, \pi(X)\pi(Y)e) = 0 ,$$
$$\pi(Z)\pi(X)\pi(Y)e = \pi([Z, X])\pi(Y)e + \pi(X)\pi(Z)\pi(Y)e ,$$

so

$$(\pi(X)\pi(Z)\pi(Y)e, \pi(X)\pi(Y)e) + (\pi([Z, X])\pi(Y)e, \pi(X)\pi(Y)e) = 0 .$$

But the second term is zero by Lemma 2.3 since $[Z, X]$ is in $\mathfrak{k}$, hence

$$\begin{aligned} 0 &= (\pi(X)\pi(Z)\pi(Y)e, \pi(X)\pi(Y)e) \\ &= -(\pi(Z)\pi(Y)e, \pi(X)\pi(X)\pi(Y)e) \\ &= -(\pi(X)\pi(Y)\pi(X)e, \pi(Z)\pi(Y)e) \\ &= (\pi(X)\pi(X)e, \pi(Y)\pi(Z)\pi(Y)e) \qquad \text{as above} \\ &= (\pi(X)\pi(X)e, \pi(Z)\pi(Y)\pi(Y)e) . \qquad \text{q.e.d.} \end{aligned}$$

Now let $\mathfrak{L}$ be the linear space of vector fields on G/K given by $\mathfrak{L} = \mathfrak{g} + E$ where E is the set of vector fields on G/K given by grad ϕ_v for $v \in E^N$ as in § 1.

Lemma 6.2. *Suppose G/K is a symmetric space, and π is a real class-one representation of G giving the immersion π: $G/K \to E^N$. If the second fundamental form is onto, then $\mathfrak{L} = \mathfrak{g} + E$ is a Lie algebra of vector fields.*

Proof. By Proposition 1 if $v \in E^N$ and $V = \text{grad}\, \phi_v$, then for any $X \in \mathfrak{g}$, $[X, V] = \text{grad}\, X\phi_v = \text{grad}\, \phi_{\pi(X)v}$. If v and $w \in E^N$, let V, W be induced vector fields on M, and $\overline{V}$, $\overline{W}$ the induced normal fields. As in Proposition 1.1 we will show that $[V, W]$ is a Killing vector for any V, W in E.

Let $X \in \mathfrak{g}$. Then

$$(\nabla_X[V, W], X) = ([X, [V, W]], X) + (\nabla_{[V,W]}X, X) .$$

Since the immersion is equivariant, we can assume that we are working at the origin of G/K, so we can take $X \in \mathfrak{p}$. Thus

$$(\nabla_X[V, W], X) = ([X, [V, W]], X) ,$$

since X is a Killing vector and $\nabla_X X = 0$

$$= ([V, [X, W]], X) - ([W, [X, V]], X)$$

$$= (V, A_{(\pi(X)w)^\perp} X) - ([X, W], A_{\bar{V}} X)$$

$$\text{(a)} \qquad = -(W, A_{(\pi(X)v)^\perp} X) + ([X, V], A_{\bar{W}} X) ,$$

(where $(\pi(X)w)^\perp$ is the normal component of $\pi(X)w$)

$$= (\pi(X)w, \alpha(X, V)) - (v, \alpha([X, W], X))$$

$$- (\pi(X)v, \alpha(X, W)) + (w, \alpha([X, V], X)) .$$

Consider the right hand side of (a) in a case by case analysis.

(i) If v and w are of the forms $\pi(Y)e$, $\pi(Z)e$, then $\alpha(X, V) = \pi(X)\pi(Y)e$, $\alpha(X, W) = \pi(X)\pi(Z)e$, for if $\{X_i\}$ is an orthonormal basis for $\mathfrak{p}$, then

$$V_0 = \sum_i (v, \pi(X_i)e)X_i ,$$

$$\alpha(X, V) = \sum_i (v, \pi(X_i)e)\pi(X)\pi(X_i)e = \pi(X)v = \pi(X)\pi(Y)e .$$

So the right hand side of (a) reduces to zero.

(ii) If v and w are of the forms $\pi(Y)\pi(Y)e$, $\pi(Z)\pi(Z)e$, then $V_0 = W_0 = 0$, and

$$\alpha([X, W], X) = \pi(X)\pi(X)\pi(Z)\pi(Z)e , \quad \alpha([X, V], X) = \pi(X)\pi(X)\pi(Y)\pi(Y)e .$$

This can be seen by using Lemma 6.1 and repeating process in (i). Also by Lemma 6.1, $\pi(X)w$ and $\pi(X)v$ have no normal components so the right hand side of (a) reduces to

$$(\pi(Z)\pi(Z)e, \pi(X)\pi(X)\pi(Y)\pi(Y)e) - (\pi(Y)\pi(Y)e, \pi(X)\pi(X)\pi(Z)\pi(Z)e)$$

$$= 0 \qquad \text{since } \pi(X)\pi(X) \text{ is symmetric.}$$

(iii) If v is of the form $\pi(Y)e$, and w is of the form $\pi(Z)\pi(Z)e$, then $w_0 = [X, V_0] = 0$, and the normal components w and v of $\pi(X)$ are both zero.

Since (i), (ii) and (iii) exhaust the possibilities, we see that $[V, W]$ is a Killing vector for any V and W in E, hence $[V, W] \in \mathfrak{g}$. But, by Proposition 1, E is stable under $\mathfrak{g}$; thus $\mathfrak{L} = \mathfrak{g} + E$ is a Lie algebra.

Remark. The Lie algebra $\mathfrak{L}$ can be constructed purely algebraically; cf. [4]. However the proof that it is in fact a Lie algebra is much more difficult.

We are now in a position to state and prove

Theorem 6.1. *Let G/K be a compact symmetric space, and π an irreducible* *real class-one representation of G giving the imbedding $\pi: G/K \to E^N$. If the second fundamental form is an onto map, and $E = \{\operatorname{grad}\phi_v \mid v \in E^N\}$, then $\mathfrak{L} = \mathfrak{g} + E^N$ is a noncompact semisimple Lie algebra with $\mathfrak{L} = \mathfrak{g} + E^N$, a Cartan decomposition. If σ is the Cartan involution, then $(\mathfrak{L}, \sigma)$ is irreducible orthogonal symmetric.*

Proof. Define a representation ρ of G on $\mathfrak{L}$ by $\rho(g)|_{\mathfrak{g}} = \operatorname{Ad}_G g$, $\rho(g)(\operatorname{grad}\phi_v) = \operatorname{grad}\phi_{\pi(g)v}$. Consider $\operatorname{ad}_{\mathfrak{L}}(\rho(g)V)$ where $V = \operatorname{grad}\phi_v$ for some $v \in E^N$. Assert $\operatorname{ad}_{\mathfrak{L}}(\rho(g)V) = \rho(g)\operatorname{ad}_{\mathfrak{L}} V\rho(g)^{-1}$. Let X be in $\mathfrak{g}$.

$$[X, \rho(g)V] = \operatorname{grad}(\phi_{\pi(X)\rho(g)v}) = \operatorname{grad}(\phi_{\pi(g)\pi(\operatorname{Ad}(g^{-1})X)v}) = \rho(g)[\operatorname{Ad}(g^{-1})X, V] ,$$

so

$$[\rho(g)V, X] = \rho(g)\operatorname{ad}_{\mathfrak{L}} V\rho(g^{-1})(X) .$$

Let $U = \operatorname{grad}\phi_u$ and $X \in \mathfrak{g}$. Then $[U, V] \in \mathfrak{g}$.

First we prove $([U, V], X) = (v, \pi(X)u)$. Let $\bar{U} = u - U$. Identifying E^N with its tangent space, as in part (i) of Lemma 6.1 we can prove $\alpha(X, V) = \pi(X)V$. So

$$\begin{aligned}([U, V], X) &= (A_{\bar{V}}U, X) - (A_{\bar{U}}V, X) = (\bar{V}, \pi(X)u) - (u, \pi(X)V)\\ &= (v, \pi(X)u) \qquad \text{by Lemma 6.1.}\end{aligned}$$

Thus

$$\begin{aligned}([\rho(g)V, U], X) &= (\pi(g)v, \pi(X)u) = (v, \pi(\operatorname{Ad} g^{-1}X)\pi(g^{-1})u)\\ &= ([V, \rho(g^{-1})U], \operatorname{Ad}(g^{-1})X) = (\operatorname{Ad}(g)[V, \rho(g^{-1})U], X) ,\end{aligned}$$

so that $\operatorname{ad}_{\mathfrak{L}}\rho(g)V(u) = \rho(g)\operatorname{ad}_{\mathfrak{L}} V\rho(g^{-1})(U)$, which gives $\operatorname{ad}_{\mathfrak{L}}\rho(g)V = \rho(g)\operatorname{ad}_{\mathfrak{L}} V\rho(g^{-1})$.

Now let $B_{\mathfrak{L}}$ be a Killing form on $\mathfrak{L}$. Then $B_{\mathfrak{L}}$ is G-invariant on E, and hence is a constant multiple of inner product on E^N (here we need the irreducibility). We now show that the above constant ≥ 0.

Consider $\operatorname{ad} e$ where $e = \operatorname{grad}\phi_e$. $\operatorname{ad} e|_k = 0$ since e is $\mathfrak{k}$-invariant.

$$([e, V], X) = (\pi(X)e, V)$$

for any $V \in E$, so $[e, V] = 0$ for all e in T_0, and

$$[e, \operatorname{grad}\phi_{\pi(X)e}] = \pi(X)e \qquad X \in \mathfrak{p} .$$

$[e, X] = \pi(X)e$, so $\operatorname{trace}(\operatorname{ad} e)^2|_{\mathfrak{p}} = n = \operatorname{trace}(\operatorname{ad} e)^2|_{T_0}$, and $B_{\mathfrak{L}}(e, e) = 2n$. Hence $B_{\mathfrak{L}}(V, V) > 0$ for any $V \in E$. If X is in $\mathfrak{g}$, then

$$B_{\mathfrak{L}}(X, X) = \operatorname{tr}(\operatorname{ad} X)^2 + \operatorname{tr}(\pi(X))^2 < 0 .$$

$\mathfrak{g}$ and E are obviously orthogonal under $B_{\mathfrak{L}}$, so $\mathfrak{L}$ is semisimple.

The map $\sigma: \mathfrak{L} \to \mathfrak{L}$ by

$$\sigma(X + V) = X - V \qquad X \in \mathfrak{g},\ V \in E$$

is obviously an involutive automorphism. The fact that $\mathfrak{L}$ is irreducible orthogonal symmetric follows from the irreducibility of the representation of $\mathfrak{g}$ on E^N and the fact that the representation is faithful.

7. Geometric results

We can now apply the above results to the problem of classifying those locally symmetric irreducible homogeneous spaces which have equivariant tight immersions. All homogeneous spaces considered are compact.

We have the situation: G/K is a locally symmetric irreducible homogeneous space, and π a nontrivial real class-one representation of G giving a 0-tight immersion $\pi: G/K \rightarrow E^N$. By corollary to Theorem 4.1 we can assume π is in fact irreducible, and we get the following classification theorems.

Theorem 7.1. *Let G/K be an irreducible locally symmetric homogeneous space, and π an irreducible real class-one orthogonal representation of G giving the immersion $G/K \rightarrow E^N$. Then the following are equivalent.*

(i) *π is 0-tight.*

(ii) *G/K is a symmetric R-space and π is in fact one of the imbeddings constructed in* [8].

(iii) *π is tight (has minimal total curvature).*

Theorem 7.2. *Let G/K be a locally symmetric homogeneous space. Then the following are equivalent.*

(i) *G/K covers a symmetric R-space.*

(ii) *There is an irreducible class-one real representation of G such that the second fundamental form of the immersion $\pi: G/K \rightarrow E^N$ is an onto map.*

Proof of Theorem 7.1. (i) $\Rightarrow$ (ii). Since π is an irreducible representation of G, the immersion $\pi: G/K \rightarrow E^N$ is substantial, and thus, since the immersion is 0-tight, Theorem 3.1 shows the second fundamental form is onto; so Theorem 6.1 shows $\mathfrak{L} = \mathfrak{g} + E$ is a semisimple Lie algebra with G the compact subgroup of Int $(\mathfrak{L})$ with Lie algebra $\mathfrak{g}$. Thus G is maximal compact in Int $(\mathfrak{L})$. Theorem 3.2 shows that π is an imbedding, so K is a subgroup of G leaving e-fixed. For suppose $H \supset K$ leaves e-fixed. Since $\pi: G/K \rightarrow E^N$ is an immersion, $\mathfrak{L}$ is the Lie algebra of $H = K$ and $\pi: G/H$ is an imbedding. $\pi: G/K \rightarrow G/H$ is a covering. Theorem 6.1 shows e has eigenvalues 0, ± 1 in $\mathfrak{L}$ and is semisimple, so by definition G/K is a symmetric R-space, and the imbedding is one of the class considered in [8].

(ii) $\Rightarrow$ (iii). Kobayashi-Takeuchi, [8, Theorem 3.1].

(iii) $\Rightarrow$ (i). See introduction to § 3.

Proof of Theorem 7.2. (i) $\Rightarrow$ (ii). Suppose $f: G/K \rightarrow M'$ is covering. Let π be the imbedding constructed in Theorem 5.2. Then $\pi_0 f$ gives required immersion.

(ii) $\Rightarrow$ (i). As above G is a maximal compact in $\mathrm{Int}(\mathfrak{L})$, $\mathfrak{L} = \mathfrak{g} + E$. Thus K is a subgroup of the isotropy group K_e of e, and both are compact and have the same Lie algebra. Hence G/K covers G/K_e which is by definition a symmetric R-space.

8. Minimal submanifolds on the sphere

Let $M^n \subset N$ be a submanifold of a Riemannian manifold. Then the mean normal curvature at a point $p \in M$ is the vector $\sum_i \alpha_p(X_i, X_i)$ where (X_i) is an orthonormal basis of M_p^n. M^n is said to be minimal in N if the mean normal curvature is zero at every point $p \in M$. We now generalize [8, Theorem 4.2].

Theorem 8.1. *Suppose G/H is an isotropy irreducible homogeneous space, and π a real class-one representation of G giving the immersion $\pi\colon G/H \to E^N$. If the second fundamental form is onto, then $\pi\colon G/H \to S^{N-1}$ is a minimal immersion.*

Proof. Let $v \in (G/H)_0$. Then A_v is a symmetric linear operator on $\mathfrak{m}$ (where $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$) given by

$$(A_v X, Y) = \tfrac{1}{2}(v, \pi(X)\pi(Y)e + \pi(Y)\pi(X)e) ,$$

where e is H-fixed. If $h \in H$, then

$$\begin{aligned}(A_{\pi(h)v}X, Y) &= \tfrac{1}{2}(\pi(h)v, \pi(X)\pi(Y)e + \pi(Y)\pi(X)e) \\ &= \tfrac{1}{2}(v, \pi(\mathrm{Ad}\, h^{-1}X)\, \mathrm{Ad}\,(h^{-1})Y + \pi(\mathrm{Ad}\, h^{-1}Y)(\mathrm{Ad}\, h^{-1}X)e) \\ &= (A_v\, \mathrm{Ad}\, h^{-1}X, \mathrm{Ad}\, h^{-1}Y) , \\ A_{\pi(h)v} &= \mathrm{Ad}\, h\, A_v\, \mathrm{Ad}\, h^{-1} .\end{aligned}$$

Now A_e is clearly H-invariant so also is A_N where

$$N = \sum \alpha(X_i, X_i) = \sum \pi(X_i)\pi(X_i)e ,$$

$\{X_i\}$ being an orthonormal basis for $\mathfrak{m}$. But there is only one symmetric linear operator on $\mathfrak{m}$ invariant under H so $A_e = A_N$ or $A_{N-e} = 0$. Since the second fundamental form is onto, $A_{N-e} = 0$ imples $N = e$. q.e.d.

The following corollary is immediate.

Corollary. *If G/H is isotropy irreducible, and π a real class-one representation of G giving a 0-tight immersion of G/H in E^N, then π gives a minimal imbedding in S^{N-1}.*

Appendix

The proof of Theorem 6.1 depends strongly on Lemma 6.1 which in turn depends on the fact that G/K is symmetric. That this is no restriction in the sense that an algebra of the form $\mathfrak{L}$ cannot be constructed for nonsymmetric isotropy irreducible spaces can be seen as follows. In [15, Theorem 1.1] Wolf proves that if $\mathfrak{L} = \mathfrak{g} + \mathfrak{p}$ is irreducible, i.e., if the action of $\mathfrak{g}$ on $\mathfrak{p}$ is irreducible, then $\mathfrak{L}$ is either a) Euclidean b) noncompact semi-simple with $\mathfrak{g} + \mathfrak{p}$ a Cartan decomposition or c) compact simple. The possibility a) is easily eliminated for geometric reasons. b) is the case studied, for then G/H would be a least covered by a symmetric R-space. If c) were true, then we would have the following situation: G/H is isotropy irreducible, G the largest connected group of isometries, and L a compact group containing G acting on G/H, so G/H has an L-invariant metric. But this metric is G-invariant, hence Lie algebra (L) = Lie algebra (G).

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UNIVERSITY OF NEW BRUNSWICK, CANADA
UNIVERSITY OF MISSOURI-ST. LOUIS